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Nonuniqueness of the Factorization Scheme in Quantum Mechanics

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We enquire into the consequences of the nonuniqueness of the factorizability of a quantum mechanical Hamiltonian in one dimension. This leads to a hierarchy of potentials, a particular class of which is endowed with the energy spectrum of the harmonic oscillator.

The algebraic properties of the factorization method have been well explored (Lahiri *et al.*, 1987). These include their connection (Gendenshtein and Krive, 1985) to supersymmetric quantum mechanics to generate partner potentials from a superpotential. The factorization method has also been widely used to determine the energy spectrum of exactly solvable potentials in quantum mechanics. However, an aspect of this scheme particularly worth noting (Mielnik, 1984; Kumar, 1987) concerns the uniqueness of the factorized expression of a quantum mechanical Hamiltonian.

Consider the following Hamiltonian, corresponding to the harmonic oscillator potential:

$$H = -\tfrac{1}{2}d^2/dx^2 + \tfrac{1}{2}x^2 \qquad (1)$$

Typically, one can resolve H into such factors as

$$H + \tfrac{1}{2} = [2^{-1/2}(d/dx + x)][2^{-1/2}(-d/dx + x) \qquad (2)$$

and introduce the corresponding annihilation and creation operators a

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and a^*,

$$a = 2^{-1/2}(d/dx + x), \qquad a^* = 2^{-1/2}(-d/dx + x) \tag{3}$$

Thus, H may be written as

$$H = aa^* - \tfrac{1}{2} \tag{4}$$

with a and a^* satisfying

$$[a, a^*] = 1 \tag{5}$$

Further,

$$Ha^* = a^*(H+1), \qquad Ha = a(H-1) \tag{6}$$

The ground-state eigenfunction of H, namely $\psi_0 = C_0 e^{-x^2/2}$ (C_0 is a constant) may be obtained from the condition $a\psi_0 = 0$, while the subsequent eigenfunctions ψ_n $(n = 1, 2, 3, \ldots)$ are determined using $\psi_n = C_n(a^*)^n\psi_0$. The latter yields the usual Hermite polynomials.

In general, the factorization of the Hamiltonian corresponding to any arbitrary potential $V(x)$, namely

$$H = -\tfrac{1}{2}\, d^2/dx^2 + V(x) \tag{7}$$

means that a and a^* are expressible in terms of the ground-state wave function ψ_0:

$$a = 2^{-1/2}[d/dx + \alpha(x)]$$

$$a^* = 2^{-1/2}[-d/dx + \alpha(x)] \tag{8}$$

where

$$\alpha(x) = d/dx\ (\ln \psi_0) \tag{9}$$

The function $a(x)$ is related to the potential $V(x)$ through

$$V(x) = \tfrac{1}{2}(d\alpha/dx + \alpha^2 - 1) \tag{10}$$

For the harmonic oscillator, $\alpha(x)$ is simply x and (10) becomes $V(x) = \frac{1}{2}x^2$.

The nonuniqueness of the representations of a and a^* stems from the fact that nothing prevents us from modifying (8) to

$$b = 2^{-1/2}[d/dx + \beta(x)]$$

$$b^* = 2^{-1/2}[-d/dx + \beta(x)] \tag{11}$$

where β is another arbitrary function of $x \neq \alpha$. The reason is that if we express H as

$$H + \tfrac{1}{2} = 2^{-1/2}(d/dx + \beta)2^{-1/2}(-d/dx + \beta) \tag{12}$$

with H still given by (7), a nontrivial solution of β emerges as

$$\beta(x) = \alpha(x) + \Phi(x) \tag{13}$$

where

$$\Phi(x) = \psi_0^{-2}\left(\text{const} + \int \psi_0^{-2}\, dx\right)^{-1} \tag{14}$$

is nonvanishing. However, to obtain $\Phi(x)$ in closed form, ψ_0 must be an inverse-square integrable function.

It is to be stressed that unlike a and a^*, the operators b and b^* do not commute to give a number

$$[b, b^*] = d\beta/dx \tag{15}$$

One consequence of this is that while bb^* is still $H+\frac{1}{2}$, i.e.,

$$bb^* = aa^* = H + \tfrac{1}{2} \tag{16}$$

the inverted product b^*b is not a constant,

$$b^*b = H - (d\beta/dx - \tfrac{1}{2}) \tag{17}$$

For the case $\alpha(x) = x$, we can express b^*b as (Mielnik, 1984; Kumar, 1987)

$$b^*b = H' - \tfrac{1}{2} \tag{18}$$

where

$$H' = -\tfrac{1}{2}\, d^2/dx^2 + V(x) \tag{19}$$

and

$$V(x) = \frac{x^2}{2} - \frac{d}{dx}\left[e^{-x^2}\left(\text{const} + \int_0^{x_0} e^{-y^2}\, dy\right)^{-1}\right] \tag{20}$$

In this way we essentially define a new Hamiltonian H' whose spectrum is identical with that of the harmonic oscillator, but whose potential is different. To see this, note that

$$H'b^* = (b^*b + \tfrac{1}{2})b^* = b^*(bb^* + \tfrac{1}{2}) = b^*(H+1) \tag{21}$$

Denoting the eigenfunctions of H' as $\Phi_n = b^*\psi_{n-1}$, with ψ_n being those of (1), it follows in a straightforward way that

$$H'\Phi_n = H'b^*\psi_{n-1} = b^*(H+1)\psi_{n-1} = b^*(n+\tfrac{1}{2})\psi_{n-1} = (n+\tfrac{1}{2})\Phi_n, \qquad n = 1, 2, \ldots \tag{22}$$

What happens if we modify (12) still further? Let us set

$$c = 2^{-1/2}[d/dx + \gamma(x)], \qquad c^* = 2^{-1/2}[d/dx + \gamma(x)] \tag{23}$$

where

$$\gamma(x) = \beta(x) + \chi(x) = \alpha(x) + \Phi(x) + \chi(x) \tag{24}$$

The function $\chi(x)$ may be determined easily by setting

$$H+\tfrac{1}{2}=2^{-1/2}(d/dx+\gamma)2^{-1/2}(-d/dx+\gamma) \tag{25}$$

one gets

$$\chi(x)=\psi_0^{-2}e^{-\int 2\Phi\,dx}\left[\text{const}+\int(\psi_0^{-2}\,e^{-\int 2\Phi\,dx})\,dx\right]^{-1} \tag{26}$$

One can evaluate the commutator $[c, c^*]$ from

$$[c, c^*]=d/dx\,(\alpha+\Phi+\chi) \tag{27}$$

Consequently,

$$c^*c=H+\tfrac{1}{2}-d/dx\,(\alpha+\Phi+\chi) \tag{28}$$

In other words, one can construct a new Hamiltonian

$$H_{\text{new}}=c^*c-\tfrac{1}{2}+d\alpha/dx=H-d/dx\,(\Phi+\chi) \tag{29}$$

having a potential $V(x)-d\Phi/dx-d\chi/dx$ and which satisfies

$$H_{\text{new}}c^*=c^*(cc^*+d\alpha/dx-\tfrac{1}{2})=c^*(H+d\alpha/dx) \tag{30}$$

Also, if ψ_n $(n=0, 1, 2, \ldots)$ are the eigenvectors of the harmonic oscillator as before, then the $\lambda_n=c^*\psi_{n-1}$ satisfy

$$\begin{aligned} H_{\text{new}}\lambda_n &= H_{\text{new}}c^*\psi_{n-1} \\ &= c^*(n-\tfrac{1}{2}+d\alpha/dx)\psi_{n-1} \\ &= (n-\tfrac{1}{2}+d\alpha/dx)\lambda_n \end{aligned} \tag{31}$$

However, it is to be noted that the coefficient of λ_n on the right-hand side of (31) is not a number. Nevertheless, when $\alpha(x)=x$, (31) gives the energy eigenvalue of the harmonic oscillator.

The orthogonality of λ_n is obvious:

$$(\lambda_j, \lambda_k)=0 \qquad \text{for} \quad j\neq k \tag{32}$$

The vector λ_0 is obtained from

$$c\lambda_0=2^{-1/2}(d/dx+\gamma)\lambda_0=0 \tag{33}$$

which implies that

$$\lambda_0=c_0\exp\left[\int_0^x \gamma(y)\,dy\right] \tag{34}$$

Further,

$$\begin{aligned} H_{\text{new}}\lambda_0 &= (c^*c-\tfrac{1}{2}+d\alpha/dx)\lambda_0 \\ &= (d\alpha/dx-\tfrac{1}{2})\lambda_0 \end{aligned} \tag{35}$$

The eigenvalue is $\frac{1}{2}$ for the harmonic oscillator potential.

To determine $\gamma(x)$ in (23), we must have a knowledge of the ground-state wave function. To illustrate this, we consider a typical form for ψ_0,

$$\psi_0(x) = \cosh^c(x) \tag{36}$$

where c (>0) is a parameter. Such a form has acquired some importance in supersymmetric quantum mechanics (Gendenshtein and Krive, 1985; Gendenshtein, 1983), due to its connection to the shape-invariant potentials. Using (36), we can immediately write down $\alpha(x)$:

$$\alpha(x) = d/dx \ln \psi_0 = c \tanh(x) \tag{37}$$

Now

$$\int dx/\psi_0^2 = \int (1-z^2)^{c-1}\, dz$$

$$= I_{c-1} \quad \text{(say)} \tag{38}$$

where $z = \tanh(x)$. Thus,

$$\Phi(x) = \cosh^{-2c}(x)(\text{const} + I_{c-1})^{-1} \tag{39}$$

where I_{c-1} satisfies the recurrence relation

$$I_c = z(1-z^2)^2/(2c+1) + 2cI_{c-1}/(2c+1) \tag{40}$$

along with

$$I_0 = z, \qquad I_1 = \tfrac{1}{2}\ln(1-z)/(1+z) \tag{41}$$

In this way $\Phi(x)$ becomes known. Once $\Phi(x)$ is determined, $\Phi(x)$ may be readily inferred from (26). For instance, let us set $c = 1$. This corresponds to

$$\psi_0 = \cosh x, \qquad \alpha(x) = \tanh x \tag{42}$$

As a result,

$$\Phi(x) = \text{sech}^2 x\, (\text{const} + \tanh x)^{-1} \tag{43}$$

Performing now the necessary integrations, we obtain an explicit form for $\chi(x)$

$$\chi(x) = \text{sech}^2 x\, (A + \tanh x)(1 + B \tanh x)]^{-1} \tag{44}$$

where A and B are constants.

To conclude, we note that we can, in general, carry out factorization of the Hamiltonian in the form

$$H + \tfrac{1}{2} = \tfrac{1}{2}[d/dx + \eta(x)][-d/dx + \eta(x)] \tag{45}$$

to *any* extent we please,

$$\eta(x) = \alpha(x) + \Phi(x) + \chi(x) + \dots \tag{46}$$

Of course, for this the ground-state wave function of the system is to be known and the functions $\Phi(x)$, $\chi(x), \ldots,$ must satisfy certain conditions of integrability. Also, whatever the forms of $\Phi(x)$, $\chi(x), \ldots,$ since relations such as those given by (31) and (35) are dependent only on $\alpha(x)$, they remain unaffected.

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